THE LEBESGUE DENSITY THEOREM

In this note, we let m denote the Lebesgue measure on $\mathbb{R}$.

Definition 1. *Let $E \subset \mathbb{R}$ be measurable, and let $x \in \mathbb{R}$. The* density *of E at x is*

$$d_E(x) := \lim_{\varepsilon \to 0} \frac{m(E \cap [x-\varepsilon, x+\varepsilon]}{2\varepsilon}$$

if the limit exists. If $d_E(x) = 1$, we say that x is a density point *for E. We let $D(E)$ denote the set of density points for E.*

A density point might not belong to the set itself: the point 0 is a density point for $E = \mathbb{R} \setminus \{0\}$.

Example 2. *Let $c \in [0,1]$; we now construct a set E such that $d_E(0) = c$.*

For any interval $I_n = [1/(n+1), 1/n]$, let $c_n \in I_n$ denote the point such that $m([1/(n+1), c_n]) = c \cdot m(I_n)$. Let

$$E = \bigcup_{n \geq 1} \left[\frac{1}{n+1}, c_n\right] \cup \left[-c_n, -\frac{1}{n+1}\right].$$

Note that, for any $n \in \mathbb{N}$, we have

$$m(E \cap [0,1/n]) = \sum_{k \geq n} m(E \cap I_k) = \sum_{k \geq n} c \cdot m(I_k) = c \cdot m([0,1/n]) = c/n.,$$

so that $m(E \cap [-1/n, 1/n]) = 2c/n$. Let us fix $\varepsilon \in (0,1)$ and let $n \in \mathbb{N}$ such that $1/(n+1) < \varepsilon \leq 1/n$. Then we have

$$\frac{m(E \cap [-\varepsilon, \varepsilon])}{2\varepsilon} \leq \frac{m(E \cap [-1/n, 1/n])}{2/n+1} = \frac{2c/n}{2/n+1} = c\frac{n+1}{n}.$$

We obtain a lower bound in a similar way

$$\frac{m(E \cap [-\varepsilon, \varepsilon])}{2\varepsilon} \geq \frac{m(E \cap [-1/(n+1), 1/(n+1)])}{2/n} = c\frac{n}{n+1}.$$

By the Sandwich Theorem, the limit

$$\lim_{\varepsilon \to 0} \frac{m(E \cap [-\varepsilon, \varepsilon])}{2\varepsilon}$$

exists and equals c.

The density function satisfies the following properties.

Proposition 3. *The following propeties hold:*

1. $d_\emptyset(x) = 0$ *and* $d_\mathbb{R}(x) = 1$,
2. $d_{E^c}(x) = 1 - d_E(x)$,
3. *if $A \subseteq B$ then $d_A(x) \leq d_B(x)$ (whenever they exist). In particular, $D(A) \subseteq D(B)$.*
4. *if $m(A \triangle B) = 0$ then $d_A(x) = d_B(x)$. In particular, $D(A) = D(B)$.*
5. $D(A \cap B) = D(A) \cap D(B)$.

Proof. The first 4 properties are left as an exercise. Let us prove (5).

From (3) we deduce $D(A \cap B) \subseteq D(A) \cap D(B)$. Let us verify the other inclusion. Let I be any interval centered at x. Since

$$I \setminus (A \cap B) = I \setminus A \cup I \setminus B,$$

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we can write

$$m(I)-m(I\cap A\cap B)\leq m(I)-m(I\cap A)+m(I)-m(I\cap B),$$

from which we obtain

$$\frac{m(I\cap A)}{m(I)}+\frac{m(I\cap B)}{m(I)}-1\leq\frac{m(I\cap A\cap B)}{m(I)}.$$

If $x\in D(A)\cap D(B)$, then the limit of the left hand side above as the diameter of I goes to 0 exists and equals 1, thus,

$$1\leq\liminf_{\varepsilon\to 0}\frac{m([x-\varepsilon,x+\varepsilon]\cap A\cap B)}{2\varepsilon}\leq 1,$$

which proves that $x\in D(A\cap B)$. □

The Lebesgue Density Theorem tells us that *almost every point is a density point*, more precisely the following result holds.

Theorem 4. *If $E\subset\mathbb{R}$ is measurable, then $m(E\triangle D(E))=0$.*

We will need the following lemma.

Lemma 5 (Rising Sun Lemma)**.** *Let $F\colon[a,b]\to\mathbb{R}$ be continuous and let $U\subseteq(a,b)$ be open. The set*

$$U_F:=\{x\in U\ :\ \text{there exists } y>x \text{ s.t. } (x,y)\subset U \text{ and } F(x)>F(y)\}$$

is open and hence can be expressed as a countable union of disjoint open intervals (a_i,b_i). Then, $F(a_i)\geq F(b_i)$.

Proof. The set U_F is open since F is continuous. Let us fix one of the intervals $(c,d)=(a_i,b_i)$ as above. We show that $F(x)\geq F(d)$ for all $x\in(c,d)$.

Let

$$s:=\max\{r\in[x,d]\ :\ F(x)\geq F(r)\},$$

and suppose that $s<d$. Then, $F(x)<F(d)$. Since $s\in[x,d)\subset U_F$, there exists $t>s$ such that $(s,t)\subset U$ and $F(s)>F(t)$.

If $t\leq d$, then $F(x)\geq F(s)>F(t)$, which contradicts the maximality of $s\in[x,d]$. If $t>d$, then $F(d)>F(x)\geq F(s)>F(t)$, which implies that $d\in U_F$, but $d\notin U_F$. Thus, $s=d$ and $F(x)\geq F(d)$. □

We can now prove Theorem 4.

Proof of Theorem 4. We start with some preliminary reductions.

(1) It is enough to show that $m(E\setminus D(E))=0$: since $D(E)\setminus E\subseteq E^c\setminus D(E^c)$, it follows that $m(D(E)\setminus E)\leq m(E^c\setminus D(E^c))=0$.

(2) Since

$$\liminf_{\varepsilon\to 0}\frac{m(E\cap[x-\varepsilon,x+\varepsilon])}{2\varepsilon}\geq\frac{1}{2}\left(\liminf_{\varepsilon\to 0}\frac{m(E\cap(x,x+\varepsilon))}{\varepsilon}+\liminf_{\varepsilon\to 0}\frac{m(E\cap(x-\varepsilon,x))}{\varepsilon}\right),$$

it is enough to show that

$$m\left(x\in E\ :\ \liminf_{\varepsilon\to 0}\frac{m(E\cap(x,x+\varepsilon))}{\varepsilon}<1\right)=0,$$

and

$$m\left(x\in E\ :\ \liminf_{\varepsilon\to 0}\frac{m(E\cap(x-\varepsilon,x))}{\varepsilon}<1\right)=0.$$

We will show the first condition, the second is proved in an analogous way by symmetry.

(3) It is enough to show that, for every $n \in \mathbb{N}$, the set
$$A_n := \left\{ x \in E \ : \ \liminf_{\varepsilon \to 0} \frac{m(E \cap (x, x+\varepsilon))}{\varepsilon} < 1 - \frac{1}{n+1} \right\}$$
has measure zero.

Let $A = A_n$ be any of the sets defined above. The function $F \colon [-n,n] \to \mathbb{R}$ defined by
$$F(x) = m(E \cap (-n,x)) - \frac{n}{n+1} x$$
is continuous. Let us also notice that for any $y > x$ we have
$$F(y) - F(x) = m(E \cap (x,y)) - \frac{n}{n+1}(y-x).$$

Fix $\varepsilon > 0$. There exists an open set U such that $A \subseteq U$ and $m(U) \leq m(A) + \varepsilon$. We claim that $A \subseteq U_F$. Indeed, if $x \in A$, there exists $\delta > 0$ sufficiently small such that $x + \delta \in U$ and
$$F(x+\delta) - F(x) = m(E \cap (x, x+\delta)) - \frac{n\delta}{n+1} < \frac{n\delta}{n+1} - \frac{n\delta}{n+1} = 0.$$
Let (a_i, b_i) be pairwise disjoint intervals such that $U_F = \cup_{i \in \mathbb{N}} (a_i, b_i)$. By the Rising Sun Lemma, $F(a_i) \geq F(b_i)$ which implies
$$m(E \cap (a_i, b_i)) \leq \frac{n}{n+1}(b_i - a_i).$$
Since $A \subseteq E$ and, as we showed, $A \subseteq U_F$, we deduce
$$\begin{aligned} m(A) &\leq \sum_{i \in \mathbb{N}} m(A \cap (a_i, b_i)) \leq \sum_{i \in \mathbb{N}} m(E \cap (a_i, b_i)) \leq \sum_{i \in \mathbb{N}} \frac{n}{n+1}(b_i - a_i) = \frac{n}{n+1} m(U_F) \\ &\leq \frac{n}{n+1} m(U) \leq \frac{n}{n+1}(m(A) + \varepsilon), \end{aligned}$$
which implies that $m(A) \leq n\varepsilon$. Since ε was arbitrary, we conclude that $m(A) = 0$, which completes the proof. □